MATH 234
EXAM 2

Student Name: Key I Student Number: ____________

Instructor: ____________ Section: ____________

Question 1. (2 points each) Answer by true or false:

1. Let A be an $m \times n$ matrix, let U be the row echelon form of A, and let F be the reduced row echelon form of A, then:
   (a) T All three matrices have the same row space.
   (b) F All three matrices have the same column space.
   (c) T All three matrices have the same null space.
   (d) T All three matrices have the same rank.
   (e) T All three matrices have the same nullity.
2. F Let V is a vector space with $\dim(V) = 4$, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in V$, then span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = V$.
3. F Let V is a vector space with $\dim(V) = n$, let S be a subspace of V, then $0 < \dim(S) < n$.
4. F If V is an infinite-dimensional vector space, then any subspace of V is infinite-dimenstional.
5. T If A is an $n \times n$ singular matrix, then $\text{rank}(A) < n$.
6. F Every linearly independent set of vectors in P_n must contain n polynomials.
7. T If V is a vector space with $\dim(V) = n$, then any $n + 1$ vectors in V are linearly dependent.
8. F If $\{v_1, v_2, \cdots, v_n\}$ are linearly independent in a vector space V, then V is finite-dimensional.
9. T If $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent in R^3, then $\exists \mathbf{x} \in R^3$ such that $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}\} = R^3$.
10. T The vectors e^x, xe^x, x are linearly independent in $C[0, 1]$.
11. F Let $f, g, h \in C^2[a, b]$, if $W[f, g, h](x) = 0$ for all $x \in [a, b]$, then f, g, h are linearly dependent in $C[a, b]$

Question 2 (3 points each) Circle the most correct answer:

1. One of the following is not a basis for P_3:

   (a) $\{1, 2x, x^2 - x\}$
   (b) $\{x - 1, x^2 + 1, x^2 - 1\}$
   (c) $\{x, x^2 + 3, x^2 - 5\}$
   (d) $\{x^2 + 1, x^2 - 1, 2\}$

2. If V is a vector space with $\dim(V) = n$, then

   (a) Any n linearly independent vectors in V span V.
   (b) Any spanning set for V must contain at most n vectors.
   (c) Any set containing less than n vectors must be linearly independent.
   (d) All of the above.

3. In P_2, the transition matrix corresponding to the change of basis from $[1, x]$ to $[x - 1, x + 1]$ is

   (a) $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
   (b) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
   (c) $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

4. $\dim(\text{span}\{1 - x, x^2, 3 + x^2, 1 + x^2\})$ equals

   (a) 0
   (b) 1
   (c) 2
   (d) 3

5. The coordinate vector of $x^2 - 2x + 4$ with respect to the ordered basis $[1 - x, 1 + x, x^2 - 2]$ is

   (a) $(-4, 2, 1)^T$
   (b) $(1, 2, -4)^T$
   (c) $(1, -2, 4)^T$
   (d) $(4, 2, 1)^T$

6. If A is a 3×4 matrix, then

(a) $\text{rank}(A) = 3$
(b) $\text{rank}(A) + \text{nullity}(A) = 3$
(c) $\text{nullity}(A) = 3$
(d) none of the above.

7. Let V be a vector space. If A is the transition matrix from an ordered basis E of V to the standard basis and B is the transition matrix from the standard basis to an ordered basis F of V, then the transition matrix from F to E is

(a) AB
(b) $B^{-1}A$
(c) AB^{-1}
(d) $A^{-1}B$

$$\begin{tikzcd} E \arrow[r, "A"] \arrow[dr, "BA"'] & S \arrow[d, "B"] \\ & F \end{tikzcd}$$

Question 3.(6 points) Let $x_1 = (2,2,2)^T$, $x_2 = (2,4,4)^T$, $x_3 = (2,6,6)^T$, $x_4 = (2,2,1)^T$, $x_5 = (2,3,2)^T$. Find a basis for R^3 from the set $\{x_1, x_2, x_3, x_4, x_5\}$. Justify your answer.

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 6 & 2 & 3 \\ 2 & 4 & 6 & 1 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$\vec{x}_1, \vec{x}_2, \vec{x}_3$

Question 4.(12 points) Consider the vector space $R^{3\times3}$, which of the following is a subspace of $R^{3\times3}$? Justify your answer, i.e., if it is a subspace then prove it satisfies the conditions, if it is not then specify which condition is not satisfied with an example.

1. $S = \{A \in R^{3\times3} \mid A^2 = A\}$.
2. $S = \{A \in R^{3\times3} \mid A \text{ is symmetric}\}$.
3. $S = \{A \in R^{3\times3} \mid A \text{ is triangular}\}$.

1) No

$I \in S$ but $2I \notin S$ since

$$(2I)^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \neq I$$

2) Yes

$S \neq \phi$ since $I \in S$

1) let $A, B \in S$ then

$(A+B)^T = A^T + B^T = A + B \Rightarrow A+B \in S$

2) let $A \in S$ & let α be a scalar

then $(\alpha A)^T = \alpha A^T = \alpha A \Rightarrow \alpha A \in S$

3) No, since $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 5 \\ 1 & 2 & 2 \end{bmatrix}$

$\notin S$

Question 5.(15 points) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 3 & 11 & -1 \\ 1 & -1 & -5 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}$.

row echelon form is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1. Find a basis for the row space of A

$\{(1,1,1,1), (0,1,3,0)\}$

2. Find a basis for the column space of A

$\{(1,-1,1,2)^T, (1,3,-1,0)^T\}$

3. Find a basis for the null space of A

let $x_3 = t$, $x_4 = s \Rightarrow \vec{x} = (2t-s, -3t, t, s)^T$

Basis: $\{(2,-3,1,0)^T, (-1,0,0,1)^T\}$

4. Find the rank and nullity of A

rank = 2

nullity = 2

Question 6.(16 points) Let $E = [x, x+2]$ and $F = [x+3, x+1]$ be two ordered bases for P_2.

1. Find the transition matrix corresponding to the change of basis from E to the standard basis $[1, x]$.

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

2. Find the transition matrix corresponding to the change of basis from the standard basis $[1, x]$ to F.

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

3. Find the transition matrix corresponding to the change of basis from E to F.

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

4. Let $p(x) = 3x - 11$, find the coordinates of p with respect to F.

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -11 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 10 \end{bmatrix}$$